Published by Institute of Physics Publishing for SISSA/ISAS
Received: *October 16, 2002*
Revised: *May 2, 2003*
Accepted: *May 19, 2003*

SUSY vs E_8 gauge theory in 11 dimensions

Jarah Evslin
INFN Sezione di Pisa
Via Buonarroti, 2, Ed. C, 56127 Pisa, Italy, and
Department of Mathematics, University of California
Berkeley, CA 94720 USA
E-mail: jarah@df.unipi.it

Hisham Sati
Michigan Center for Theoretical Physics, Randall Laboratory, University of Michigan
Ann Arbor, MI 48109 USA
E-mail: hsati@umich.edu

Abstract: Diaconescu, Moore and Witten have shown that the topological part of the M-theory partition function is an invariant of an E_8 gauge bundle over the 11-dimensional bulk. Any construction of 11d SUGRA from gauge bundle data must satisfy a number of constraints in order to correctly reproduce the the known 10-dimensional physics on each boundary component. We analyse these constraints and in particular use them to attempt an approximate construction of the 11d gravitino as a condensate of the gauge theory fields.

Keywords: Field Theories in Higher Dimensions, Solitons Monopoles and Instantons, M-Theory.

Contents

1. The motivation

Six years ago Hořava and Witten demonstrated [1] that when M-theory is compactified on a manifold with boundary, the anomalies caused by chiral gauginos and gravitinos on each boundary component precisely cancel the anomalies that flow in from bulk. This cancellation occurs only if each boundary component supports precisely 248 10-dimensional vectormultiplets, all transforming in the adjoint representation of E_8. Furthermore as explained in refs. [2, 3], the topological contribution to the M-theory partition function is in fact an invariant of the Dirac operator of a mysterious *11-dimensional* E_8 gauge bundle.

While the nature of this gauge theory is entirely unknown, the delicate anomaly-cancellation of Hořava and Witten as well as the 10-dimensional $N = 1$ supersymmetry on every boundary component place strong constraints on its construction. We feel that the analysis of these constraints is a necessary first step in an attempt to understand the gauge theory.

In this paper we will propose the simplest possible particle content of such a proposal and then apply the above constraints. In particular we will combine a constraint on the four-form, 10d SUSY covariance and 11d Lorentz covariance to construct the supergravity fields from the gauge fields. We will then try to understand how this construction can be consistent with 11d supersymmetry. Such proposals have been considered previously in refs. [4]–[7]. An apparently unrelated proposal which sacrifices the compactness of the E_8 but preserves supersymmetry has appeared in refs. [8, 9]. While preserving the supersymmetry is an extraordinary advantage, it is not known whether the noncompactness of this E_8 would then lead to a noncompact gauge group for the heterotic string. While providing a fascinating alternative to the class of gauge theories considered in this note, further speculation along these lines will be deferred to a sequel.

For simplicity we will often restrict our attention to the case of flat, topologically trivial 11-dimensional space, although we generalize our results to curved space in Subsection 3.3. We will also systematically neglect higher order Fermi field contributions.

It was shown in ref. [10] that E_8 gauge invariance combined with local supersymmetry invariance requires the relation

$$\frac{G_4}{2\pi} = \frac{1}{16\pi^2}\,\mathrm{tr}\left(F \wedge F + \frac{1}{2} R \wedge R\right) \tag{1.1}$$

between the 11d 4-form field strength G_4 and the 10D $N = 1$ vectormultiplet's fieldstrength F on every 10-dimensional boundary component.[1] Following [3] we consider an E_8 gauge bundle such that eq. (1.1) holds everywhere in the 11-dimensional bulk. The fact that such a bundle exists is a consequence of M-theory's shifted flux quantization condition [2]. The uniqueness of this bundle results from the uniquely simple low dimensional topology of the E_8 group manifold.

In addition to the above 248 gauge bosons, we consider 248 adjoint[2] Majorana fermions also propagating in the 11d bulk. For now we also include an 11d graviton, although it is possible that the graviton field is in fact a composite of the other gauge theory fields. Using eq. (1.1) we can construct the 11d SUGRA 4-form G_4 from the vectors. Ten dimensional $N = 1$ SUSY covariance allows us to find the analogous construction of a chiral half of the 11-dimensional gravitino[3] up to a mysterious problem related to the fact that we do not understand the role of the graviton in this story. Eleven-dimensional Lorentz invariance allows us to construct the other half. Thus far, each gauge theory configuration is identified with a single SUGRA configuration, meaning that the construction cannot be covariant under 11d SUSY as the gauge fields are not part of any representation of 11d SUSY.

To remedy this we identify each gauge field configuration with not only the single SUGRA configuration given earlier, but with all of the SUGRA configurations which are related to that configuration by an 11d SUSY transformation. Thus SUGRA field configurations related by SUSY transformations will be identified with the same gauge field configuration and thus the same physical state. It is a critical check of the consistency of this construction that physically equivalent configurations on the gauge theory side are also equivalent on the SUGRA side, and in fact E_8 gauge transformations are realized as abelian gauge transformations of the M-theory 3-form.

In section 2 we review the standard arguments for an E_8 gauge theory in the bulk. In section 3 we present our construction for the bulk gravitino in terms of gauge theory fields and show that this construction is consistent with 10 and 11-dimensional supersymmetries.

[1] In fact there is a choice of boundary conditions on each boundary component [10, 11], which is a choice of chirality of the 10d fermions. The two chiralities couple to the 10d E_8 bundle and to its mirror image [12].

[2] We remind the reader that the adjoint and fundamental representations of E_8 are isomorphic.

[3] Independent of conjectures about mysterious bulk gauge theories, we expect this relation of the gravitino to the gauge theory fields to hold on the boundary.

JHEP05(2003)048

In section 4 we review efforts to construct a quantum theory. The form of such a gauge theory is far from obvious, as the usual kinetic term $F \wedge *F$ would not be consistent with renormalizability. We then describe calculations to which the E_8 gauge bundle formalism may be applied, in an attempt to justify the persuit of such an elusive quantum theory. We conclude with some remarks on SUSY breaking, the graviton and also a relation to other E_8's in the final section.

2. E_8 gauge theory

2.1 Why an E_8 bundle?

The low energy effective description of M-theory is 11-dimensional supergravity [13]. The fields of this theory live in a single supermultiplet which contains the graviton, the gravitino ψ and a three-form C_3 whose exterior derivative (times 6) is the four-form fieldstrength G_4. If the dynamics of M-theory are to be formulated in terms of an E_8 gauge theory, it would be useful to have explicit relations between the fields of the 11d supermultiplet and the data of the gauge bundle: the 1-form connection A with fieldstrength F and an adjoint Majorana "gaugino" χ.

The conjectured relations arise from the synthesis of several observations. First, in ref. [1] it is shown that gauge and gravitational anomaly cancellation on any 10-dimensional boundary of M-theory enforces the relation (1.1) on the boundary, where R is the curvature two-form (of the tangent bundle). In ref. [2] Witten used locality to argue that such relations, at the level of cohomology, can be extended to the bulk and moreover that such an extension uniquely specifies an E_8 bundle. Note that it does not follow from this argument that the E_8 curvature itself plays any kind of a dynamical role in the theory.

One reason[4] that one may suspect that the E_8 gauge fieldstrength appears in a bulk theory is as follows. The low energy effective action for M-theory on the 11-fold Y^{11} contains the topological terms

$$I = 2\pi \int_{Y^{11}} C_3 \wedge (G_4 \wedge G_4 - I_8) \tag{2.1}$$

where I_8 is a quartic in the curvature tensor. Using a result from ref. [10] this can be related [2] to a sum of indices of an E_8 gauge theory on an auxilliary 12-dimensional manifold.[5]

In ref. [3] a theorem of Atiyah, Patodi and Singer [14] was used to explicitly evaluate the contribution of this topological term and the pfaffian determinant of the Rarita-Schwinger operator to the phase of the path integral measure:

$$\Phi = Pf(D_{RS})e^{i\int I} = |Pf(D_{RS})| \exp(\frac{2\pi i}{4}\left((h_{E_8}+\eta_{E_8}) + \frac{2\pi i}{8}(h_{RS}+\eta_{RS})\right). \tag{2.2}$$

[4] Another very different reason has appeared in [7].

[5] More precisely, the ambiguity in I is the integral of its exterior derivative over a closed 12-manifold. This integral may be nonvanishing because C_3 is not necessarily globally defined. The path integral measure is well defined if this integral, added to a contribution from the square root of the determinant of the Rarita-Schwinger operator, is an integer. It was shown in ref. [10] that the integral is in fact a sum of indices from an E_8 gauge theory and so is guaranteed to be an integer.

JHEP05(2003)048

Here η is the η-invariant of the corresponding operator (the E_8 gauge theory Dirac operator and then the Rarita-Schwinger operator) while h is its number of zeromodes. Thus a part of the path integral measure of 11-dimensional supergravity can be reexpressed in terms of a mysterious bulk E_8 gauge theory. Furthermore it was shown that the partition function consists of a sum over E_8 gauge field configurations. In fact, the utility of classifying configurations by $Tr(F^2)$ rather than G_4 predates the realization that $Tr(F^2)$ is an E_8 bundle invariant [15].

The fact that one factor in the M-theory partition function is the index of fermions charged under an E_8 gauge symmetry does not prove that there actually are fermions charged under an E_8, but the goal of the present paper is to understand how the existence of such fermions, and such a gauge symmetry, could be consistent with what we know of SUSY in 11 dimensions.

If there is such an E_8 gauge symmetry in the 11-dimensional bulk, a natural guess for its relation to the four-form fieldstrength is simply eq. (1.1). The rest of this paper will be an investigation of the consequences of this guess. The corresponding relation between the gravitino and gauginos will appear in section 3.

2.2 E_8 bundles and solitons

If there is such a gauge theoretic description of low energy M-theory, it must be shown that E_8 gauge theory correctly reproduces the M-theory soliton spectrum [6]. To compute the gauge theory's soliton spectrum we will need to review the topology of the group manifold E_8.

The low dimensional topology of E_8 is in one way the simplest among nonabelian Lie groups. E_8 has only one nontrivial homotopy group of dimension less than 15, which is $\pi_3(E_8) = \mathbb{Z}$. This means that on a manifold of dimension less than 16, E_8 bundles are topologically characterized by a single characteristic class, the first Pontrjagin class

$$p_1 = \frac{\mathrm{Tr}(F \wedge F)}{8\pi^2}\,. \tag{2.3}$$

The only restriction on this class is that its integral over any 4-cycle be an even integer. All other semisimple Lie groups have additional nontrivial low dimensional homotopy groups and therefore their principal bundles cannot be completely characterized by a single characteristic class.

This agrees beautifully with what we know of M-theory, which at low energies is also described by a 4-form. In fact substituting (2.3) into (1.1) we learn that the 4-form flux of M-theory is a combination of this characteristic class and the first Pontrjagin class of the tangent bundle

$$\frac{G_4}{2\pi} = \frac{p_1(E_8)}{2} + \frac{p_1(TM)}{4}\,. \tag{2.4}$$

Notice that the shifted flux quantization condition [2] of G_4 is automatic in this construction. The first term on the right hand side is an integral cohomology class, while the second may be an integral cohomology class or may be half[6] of an integral cohomology class. Therefore the failure of the left hand side to be integral is precisely equal to the failure of the second term on the right hand side, that is, the *mod* 2 part of $p_1(TM)/2$.

As a result of the fact that an E_8 bundle is described by a single closed form, an E_8 bundle on a manifold of dimension less than 16 has only one kind of topological defect, the M5-brane. This is the codimension 5 defect where the form fails to be closed. If the soliton spectrum contains the M5-brane then it automatically contains the M2-brane. For example, an M2-brane is created when two M5-branes cross via the Hanany-Witten mechanism [16], the M5-branes can usually be moved off to infinity. Alternately, an M2-brane may be constructed as a limit of M5-branes that wrap a trivial 3-cycle supporting C flux as integrated on a coordinate patch that contains the entire trivial cycle. Such M5-branes are dielectric M2-branes [17] and as the three-cycle shrinks to zero size become ordinary M2-branes.

String theory on backgrounds in which such a $(5+1)$-dimensional defect is linked by either a 4-sphere or $\mathbf{RP}^4$ have been studied extensively and in particular their soliton spectra are known. We will now recover the 5-brane spectrum as the spectrum of E_8 defects. We will classify these defects by the restriction of their E_8 bundles to the 4-manifolds that link them. If this link is an S^4 then the bundle can be trivialized on the northern and southern hemispheres and the transition function on the 3-sphere equator must be an element $n \in \pi_3(E_8) = \mathbb{Z}$. Likewise the bundle can be trivialized on the only hemisphere of an $\mathbf{RP}^4$ and the transition maps its equatorial $\mathbf{RP}^3$ to E_8. These maps can be constructed by considering maps from S^3 to E_8, which are classified by $\pi_3(E_8) = \mathbb{Z}$, and then filtering them through $\mathbf{RP}^3$. Only the maps corresponding to even integers can be filtered through $\mathbf{RP}^3$ and so the maps from $\mathbf{RP}^3$ to E_8 are classified by even elements $2n \in 2\pi_3(E_8) = 2\mathbb{Z}$. $\mathbf{RP}^4$ is not orientable, and so a quarter of p_1 of its tangent bundle is not well-defined. It is suggested in [2] that the shift is quantified by the fourth Stiefel-Whitney class w_4

$$\int_{S^4} \frac{G_4}{2\pi} = \int_{S^4} \frac{p_1(\mathrm{E}_8)}{2} + \int_{S^4} \frac{p_1(\mathrm{TM})}{4} = \frac{2n}{2} + \frac{0}{4} = n \tag{2.5a}$$

$$\int_{\mathbf{RP}^4} \frac{G_4}{2\pi} = \int_{\mathbf{RP}^4} \frac{p_1(\mathrm{E}_8)}{2} + \int_{\mathbf{RP}^4} w_4(\mathrm{TM}) = \frac{4n/2}{2} + \frac{1}{2} = n + \frac{1}{2}\,. \tag{2.5b}$$

The lift to integral cohomology of the Stiefel-Whitney class is not canonical, but different choices may be absorbed into a shift in n. The physics does appear to be sensitive to the total of these two contributions, as it determines the rank of the worldvolume gauge group in a IIA reduction. By Gauss' Law these integrals are equal to the total M5-brane charge linked by the 4-cycle over which the integral is performed. Thus the first

[6] By half of an integral cohomology class ω we mean consider the image of ω in the cohomology map induced by multiplication of the coefficient ring by 2 and then divide the answer by two. If there is Z_{2k} torsion, then division by 2 is not well defined and so one needs a prescription for which quotient to take. It was conjectured in [2] that the correct prescription is to force the answer to agree with the 4th Stieffel-Whitney class.

JHEP05(2003)048

configuration describes n M5-branes, while the second describes an OM5 plane which carries $n+1/2$ units of M5-brane charge. Recalling [15] that OM5 planes always carry half integer charge we see that the spectrum of M5-brane charges is correctly reproduced by E_8 gauge theory.

3. The construction

3.1 Constructing the supergravity fields

Before relating the SUGRA and gauge theory fields, we will take a moment to review 10 and 11-dimensional $N=1$ SUGRA and to establish our conventions. The 11-dimensional supermultiplet consists of an elfbein e, a gravitino Ψ and a 3-form gauge potential C. The transformations of these fields under 11-dimensional supersymmetry transformations are as follows [13]

$$\delta e_A{}^m = \frac{1}{2}\bar{\eta}\Gamma^m\Psi_A \tag{3.1a}$$

$$\delta C_{ABC} = -\frac{\sqrt{2}}{8}\bar{\eta}\Gamma_{[AB}\Psi_{C]} \tag{3.1b}$$

$$\delta\Psi_A = D_A\eta + \frac{\sqrt{2}}{288}\left(\Gamma_A{}^{BCDE} - 8\delta_A^B\Gamma^{CDE}\right)\eta G_{BCDE} \tag{3.1c}$$

where η is the 32-component Majorana spinor that parameterizes the variation.

The 10-dimensional vector supermultiplets [10] , which propagate on the boundary M^{10}, consist of the E_8 gauge field A (with field strength $F_{CD} = \partial_C A_D - \partial_D A_C + [A_C, A_D]$) and spin 1/2 Majorana-Weyl fermions (gluinos) χ in the adjoint representation, obeying $\Gamma_{11}\chi = \chi$. Their supersymmetry transformation laws are

$$\delta A_A^i = \frac{1}{2}\bar{\eta}\Gamma_A\chi^i \tag{3.2a}$$

$$\delta\chi^i = -\frac{1}{4}\Gamma^{AB}F_{AB}^i\eta \tag{3.2b}$$

with spacetime indices $A, B = 0, \ldots, 9$ in an orthonormal frame, and E_8 gauge group indices $i = 1, \ldots, 248$, and where η is the 16-component Majorana-Weyl spinor that parameterizes the transformation.

With these conventions established we may finally construct the gravitino from the gauge fields. It follows from the construction (1.1) for the 4-form G_4 that for some E_8 gauge choice or equivalently a gauge choice for C_3:

$$C_{ABC} = \frac{1}{24\pi}\left(A_{[A}^i\partial_B A_{C]}^i + i\frac{2}{3}f^{ijk}A_{[A}^i A_B^j A_{C]}^k\right). \tag{3.3}$$

Consider this relation restricted to a 10-dimensional boundary. Although there is no bulk supersymmetry, the boundary theory is that of ref. [10] and so enjoys 10-dimensional $N=1$ supersymmetry. Performing a rigid $N=1$ SUSY transformation on both sides we arrive at an expression for the gravitino

$$\Gamma_{[AB}\Psi_{C]} = -\frac{\sqrt{2}}{12\pi}\Gamma_{[A}F_{BC]}^i\chi^i. \tag{3.4}$$

where we have dropped pure gauge terms of the form $\partial_B(A_A\bar{\eta}\Gamma_C\chi)$. Contracting with Γ^{ABC} gives

$$\not{\Psi} = \frac{\sqrt{2}}{12\pi}\cdot\frac{1}{d-1}\Gamma^{AB}F^i_{AB}\chi^i\,. \tag{3.5}$$

On the other hand, contracting with Γ^{AB}, and using (3.5), we find

$$\Psi_C = -\frac{\sqrt{2}}{12\pi}\cdot\frac{1}{(d-1)}\left[a\Gamma^{AB}F^i_{AB}\Gamma_C + b\Gamma^B F^i_{BC}\right]\chi^i\,. \tag{3.6}$$

where $a = 1/(d-2)$ and $b = 2(d-3)/(d-2)$.

This is a disturbing result, as a 10-dimensional supersymmetry transformation on the right hand side does *not* yield (3.1c). First, it misses the derivative of η. This is not a problem as we have only used rigid supersymmetry transformations and so this term vanishes. However the variation of Ψ produces an unwanted term, proportional to the kinetic term $F_{AB}F^{AB}$, that cannot so easily be dismissed. The origin of the term is as follows. The expression for C_{ABC} is a wedge product of forms, but when we take the SUSY transformation on A the one form changes to a zero form, χ. The SUSY variation of this χ yields an F^2 term that has two new dummy indices. Due to the structure of the spinor indices, these necessarily right-multiply the gamma matrices and so are not antisymmetrized with the indices of the other gamma matrix. Thus $\delta\Psi$ contains a term with two F's that are contracted, in stark contrast with the 11d SUSY transformation which yields a totally antisymmetrized four-form.

This may suggest that (3.3) is not in fact SUSY covariant. One might think this could be remedied by the addition of a closed form to the right hand side, however were such an addition required it would break the abelian gauge invariance of 11d SUGRA. Instead one is therefore led to the conclusion that the original constraint, (1.1), is not SUSY covariant, even on a 10d boundary. Thus it appears as though another term would need to be added to the constraint to impose SUSY covariance. The constraint on any such term is that it play the same role as the constraint in cancelling eq. (2.9) of ref. [1]. Such an additional term, if we impose that it be Ψ-independent, appears not to exist.

One possible pessimistic conclusion is that the constraint is simply not at all SUSY covariant and so cannot be used to glean information about the gravitino via SUSY transformations. However the problematic terms involve the graviton, whose role in this story and in particular whose relation to the gauge theory is as yet entirely mysterious. Therefore one may interpret this apparent failure of covariance as a puzzle whose resolution places a very strong, if not lethal, constraint on the role that the graviton must play. Another possibility is that the covariance has been destroyed by our truncations, and that were we to consider the curvature corrections and the higher order Fermi terms the covariance would be restored. Below we will see that indeed the inclusion of curvature terms dramatically alters the form of this construction.

We are finally ready to extend our results to 11 dimensions. We choose the gauge fields in the bulk so that these same relations hold. However the 11-dimensional Lorentz group has no Majorana-Weyl representation, and so Lorentz invariance forces us to reinterpret the fermions in this construction as Majorana fermions. More generally there may be

JHEP05(2003)048

additional terms which vanish when the Weyl condition is imposed, to determine such terms one must impose SUGRA covariance. To recover the above construction on the boundary, we impose the boundary conditions

$$A_{11} = 0\,, \qquad \Gamma_{11}\chi = \chi \tag{3.7}$$

where the 11 direction is taken to be perpendicular to the boundary.

3.2 SUSY transformations

So far we have a configuration of SUGRA fields (C, Ψ) for every configuration of gauge fields (A, χ). However our constructions (3.3) and (3.6) are not covariant under 11d SUSY transformations because the l.h.s. transforms while the r.h.s. is not in any 11d SUSY representation. To attain covariance we will identify the entire gauge and SUSY orbit $(C', \Psi') \sim (C, \Psi)$ with each gauge orbit of (A, χ). We will now use the 11-dimensional SUSY transformations of the SUGRA fields to find constructions for all (C', Ψ') related to the unprimed fields by a single SUSY transformation with a small Majorana spinor parameter η.

For the three-form we obtain

$$\begin{aligned}
\frac{1}{24\pi}\left(A^i_{[A}\partial_B A^i_{C]} + i\frac{2}{3}f^{ijk}A^i_{[A}A^j_B A^k_{C]}\right) &= C_{ABC} = C'_{ABC} - \delta_\eta C_{ABC} \\
&= C'_{ABC} + \frac{\sqrt{2}}{8}\overline{\eta}\Gamma_{[AB}\Psi_{C]} \\
&= C'_{ABC} - \frac{1}{48\pi}\overline{\eta}\Gamma_{[A}F^i_{BC]}\chi^i
\end{aligned} \tag{3.8}$$

and similarly for the gravitino

$$\begin{aligned}
-\frac{\sqrt{2}}{12\pi}\Gamma_{[A}F^i_{BC]}\chi^i &= \Gamma_{[AB}\Psi_{C]} = \Gamma_{[AB}\Psi'_{C]} - \delta_\eta\Gamma_{[AB}\Psi_{C]} \\
&= \Gamma_{[AB}\Psi'_{C]} - \Gamma_{[AB}D_{C]}\eta - \frac{\sqrt{2}}{288}\gamma^{DEFG}_{ABC}\eta G_{DEFG} \\
&= \Gamma_{[AB}\Psi'_{C]} - \Gamma_{[AB}D_{C]}\eta - \frac{\sqrt{2}}{288}\cdot\frac{1}{8\pi}\gamma^{DEFG}_{ABC}\eta F_{DE}F^i_{FG}
\end{aligned} \tag{3.9}$$

where we have defined

$$\gamma^{DEFG}_{ABC} = \Gamma_{ABC}\Gamma^{DEFG} - 8\delta^D_{[C}\Gamma_{AB]}\Gamma^{EFG}\,. \tag{3.10}$$

Assembling these results the solution for (C', Ψ') is then

$$C'_{ABC} = \frac{1}{24\pi}\left(A^i_{[A}\partial_B A^i_{C]} + i\ frac23 f^{ijk}A^i_{[A}A^j_B A^k_{C]}\right) + \frac{1}{48\pi}\overline{\eta}\Gamma_{[A}F^i_{BC]}\chi^i \tag{3.11a}$$

$$\Gamma_{[AB}\Psi'_{C]} = -\frac{\sqrt{2}}{12\pi}\Gamma_{[A}F^i_{BC]}\chi^i + \Gamma_{[AB}D_{C]}\eta + \frac{\sqrt{2}}{288}\cdot\frac{1}{8\pi}\gamma^{DEFG}_{ABC}\eta F^i_{DE}F^i_{FG}\,. \tag{3.11b}$$

One can check that restricted to a 10-dimensional slice with η Majorana-Weyl this transformation has the same effect on the right hand sides of the equations as a 10d SUSY transform on A and χ, and so reduces correctly to the Hořava-Witten case.

By construction, the 11d supersymmetry algebra is still satisfied in this proposal. For example, applying the above transformations twice on the 3-form C one finds

$$\begin{aligned}
C''_{ABC} &= C'_{ABC} + \delta_{\eta'} C'_{ABC} = C'_{ABC} + \delta_{\eta'} C_{ABC} + \delta_{\eta'}\delta_\eta C_{ABC} \\
&= \frac{1}{24\pi}\left(A^i_{[A}\partial_B A^i_{C]} + i\frac{2}{3} f^{ijk} A^i_{[A} A^j_B A^k_{C]}\right) + \frac{1}{48\pi}(\overline{\eta}+\overline{\eta}')\Gamma_{[A} F^i_{BC]}\chi^i - \\
&\quad - \frac{\sqrt{2}}{8}\overline{\eta}\Gamma_{[AB} D_{C]}\eta' - \frac{1}{1152}\cdot\frac{1}{8\pi}\overline{\eta}\gamma^{DEFG}_{ABC}\eta' F^i_{DE} F^i_{FG}
\end{aligned} \tag{3.12}$$

and the second variation of C is

$$\delta_{\eta'}\delta_\eta C_{ABC} = -\frac{\sqrt{2}}{8}\overline{\eta}\Gamma_{[AB} D_{C]}\eta' - \frac{1}{1152}\cdot\frac{1}{8\pi}\overline{\eta}\gamma^{DEFG}_{ABC}\eta' F^i_{DE} F^i_{FG}\,. \tag{3.13}$$

The commutator of two such variations is[7]

$$[\delta_{\eta'}, \delta_\eta] C_{ABC} = -\frac{1}{3}\overline{\eta}\Gamma^D \eta' G_{ABCD} \tag{3.15}$$

reproducing the Poincaré supersymmetry algebra in flat space [13], up to pure gauge terms of the form $\partial_{[A}\Lambda_{BC]}$, with $\Lambda_{BC} = C_{BCD}\overline{\eta}\Gamma^D\eta'$.

3.3 Curvature corrections

We now describe the full construction, including the term $Tr(R\wedge R)/2$ in G_4 although not including higher order fermion corrections. We begin with the variation of the Riemann tensor with respect to the metric using the Palatini formula [8]

$$\delta R^A{}_{BCD} = \nabla_C\left(\delta\tilde{\Gamma}^A{}_{BD}\right) - \nabla_D\left(\delta\tilde{\Gamma}^A{}_{BC}\right). \tag{3.16}$$

The variation of the Christoffel symbols with respect to the metric is

$$\delta\tilde{\Gamma}^A{}_{BC} = \frac{1}{2} g^{AE}[\partial_C(\delta g_{BE}) + \partial_B(\delta g_{CE}) - \partial_E(\delta g_{BC})]\,. \tag{3.17}$$

Using the supersymmetry variation of the metric, one obtains

$$\delta R^A{}_{BCD} = \frac{1}{2} g^{AE}\left[\nabla_C[\partial_D(\overline{\eta}\Gamma_{(B}\Psi_{E)}) + \partial_B(\overline{\eta}\Gamma_{(D}\Psi_{E)}) - \partial_E(\overline{\eta}\Gamma_{(B}\Psi_{D)})] - [C\leftrightarrow D]\right] \tag{3.18}$$

Finally we contract with another R to find the desired

$$\delta Tr(R\wedge R) = \left[\nabla_C[\partial_D(\overline{\eta}\Gamma_{(B}\Psi_{A)}) + \partial_B(\overline{\eta}\Gamma_{(D}\Psi_{A)}) - \partial_A(\overline{\eta}\Gamma_{(B}\Psi_{D)})] - [C\leftrightarrow D]\right] R^{ABCD} \tag{3.19}$$

where it is understood that the indices of Γ are symmetrized with those of Ψ. The full construction of $\Gamma\Gamma\Psi$ is then simply the old construction plus an inverse exterior derivative of the right hand side of (3.19).

[7] A useful property here is the Majorana flip in 11 dimensions [23]

$$\left(\bar{\lambda}\Gamma^{I_1}\Gamma^{I_2}\ldots\Gamma^{I_n}\eta\right) = (-1)^n\left(\bar{\eta}\Gamma^{I_1}\Gamma^{I_2}\ldots\Gamma^{I_n}\lambda\right).$$

[8] We denote the Christoffel symbols by $\tilde{\Gamma}$ to avoid confusion with the Gamma matrices.

More explicitly, the M-theory 3-form is

$$C_3 = C.S.^{(\text{gauge})} + C.S.^{(\text{grav.})} \tag{3.20}$$

where

$$C.S.^{(\text{grav.})} = \frac{1}{48\pi}\epsilon^{ABC}\left[\tilde{\Gamma}^{D}_{AE}\partial_B\tilde{\Gamma}^{E}_{CD} + \frac{2}{3}\tilde{\Gamma}^{F}_{AD}\tilde{\Gamma}^{D}_{BG}\tilde{\Gamma}^{G}_{CF}\right]. \tag{3.21}$$

The curvature-corrected construction of the gravitino is then

$$-\frac{\sqrt{2}}{8}\bar{\eta}\Gamma_{[AB}\Psi_{C]} = \frac{1}{48\pi}\Gamma_A F^{i}_{BC}\chi^{i} + \delta C.S.^{(\text{grav.})} \tag{3.22}$$

where the variation of the $C.S.^{(grav.)}$ is

$$\begin{aligned}
\delta C.S.^{(\text{grav.})} = \frac{1}{48\pi}\cdot\frac{1}{4}\epsilon^{ABC}\Big[& g^{DH}\left[\partial_E(\bar{\eta}\Gamma_{(A}\Psi_{H)}) + \partial_A(\bar{\eta}\Gamma_{(E}\Psi_{H)}) - \partial_H(\bar{\eta}\Gamma_{(A}\Psi_{E)})\right]\times \\
& \times\partial_B\tilde{\Gamma}^{E}_{CD} + \tilde{\Gamma}^{D}_{AE}\partial_B[g^{EH}\left[\partial_C(\bar{\eta}\Gamma_{(D}\Psi_{H)}) + \partial_D(\bar{\eta}\Gamma_{(C}\Psi_{H)}) - \right. \\
& \left. - \partial_H(\bar{\eta}\Gamma_{(C}\Psi_{D)})\right] + \\
& + \frac{2}{3}g^{FH}\left[\partial_A(\bar{\eta}\Gamma_{(D}\Psi_{H)}) + \partial_D(\bar{\eta}\Gamma_{(A}\Psi_{H)}) - \partial_H(\bar{\eta}\Gamma_{(A}\Psi_{D)})\right]\times \\
& \times\tilde{\Gamma}^{D}_{BG}\tilde{\Gamma}^{G}_{CF} + \frac{2}{3}\tilde{\Gamma}^{F}_{AD}g^{DH}\left[\partial_B(\bar{\eta}\Gamma_{(G}\Psi_{H)}) + \partial_G(\bar{\eta}\Gamma_{(B}\Psi_{H)}) - \right. \\
& \left. - \partial_H(\bar{\eta}\Gamma_{(B}\Psi_{G)})\right]\tilde{\Gamma}^{G}_{CF} + \\
& + \frac{2}{3}\tilde{\Gamma}^{F}_{AD}\tilde{\Gamma}^{D}_{BG}g^{GH}\left[\partial_C(\bar{\eta}\Gamma_{(F}\Psi_{H)}) + \partial_F(\bar{\eta}\Gamma_{(C}\Psi_{H)}) - \right. \\
& \left. - \partial_H(\bar{\eta}\Gamma_{(C}\Psi_{F)})\right]\Big].
\end{aligned} \tag{3.23}$$

This is a first order linear differential equation for the construction of the gravitino Ψ. Notice that even on a 10-dimensional boundary, when curvature terms are not omitted, the 10-dimensional SUSY variation of (1.1) yields only a differential equation for the restriction of Ψ to the boundary because derivatives of Ψ appear in the SUSY variation of the curvature.

The above analysis could have equivalently been done using vielbeins and spin connections. Given an expression of the metric in terms of the gauge fields, an explicit one for the Christoffel symbols and the Riemann tensor could be constructed. This will be left to future work.

4. Possible applications

The existence of a quantum M-theory has always required a leap of faith, and if one denies its supersymmetry this leap is yet more severe. The only motivation for such a leap can be the existence of applications. There must be calculations that the E_8 gauge theory formulation allows one to solve more easily than other approaches.

If M-theory configurations are classified by E_8 gauge bundles over 11d, then IIA configurations must be classified by LE_8 bundles over 10d. It has been conjectured that RR fluxes in IIA are classified by twisted K-theory, however the E_8 approach must yield not only all RR fluxes but also all NS fluxes, and thus yield a structure which for each allowed value of

JHEP05(2003)048

H reduces to the corresponding twisted K-theory. This structure can be found explicitly, by classifying LE_8 bundles. Partial results have already appeared. For example, the fact that the D6-brane must wrap a $spin^c$ submanifold (which implies that it corresponds to a K-homology class) has been shown to be the obstruction to the existence of the fibration of the central extension of LE_8 (the M-theory circle) [24]. In addition the Freed-Witten anomaly on a D8-brane worldvolume has been shown to be an obstruction to the existence of the corresponding LE_8 bundle in ref. [25], and moreover the D6-brane insertions which cancel the anomaly are precisely the defects that cancel the topological obstruction. The general strategy for showing that all of the Freed-Witten anomalies are obstructions to the existence of this LE_8 bundle has been outlined in refs. [6, 26]. However the E_8 formalism should allow one to compute the analog of the Freed-Witten anomaly for NS5-branes as well.

A second, but less concrete, proposed application is as follows. The E_8 bundle formalism naturally assigns maps from the worldvolumes of M2 and M5 branes to E_8 [26]. In particular, the self-dual 3-form T on the M5-brane worldvolume consists of the jacobian determinants of 3×3 submatrices of this map. One may use this map to embed the M2 and M5-branes in the 259-dimensional total space, and search for a worldvolume action in terms of 259 bosonic worldvolume fields which reproduces the known worldvolume action of the M2 and M5-branes in 11-dimensions. This appears promising as, for example, the T term in the M2-brane action is the jacobian term that one would expect from such a dimensional reduction. J. Plefka has suggested that, as a further test, one could then use the diffeomorphism invariance of the membrane in the 259-dimensional theory to show that an E_8 gauge theory must inhabit an end-of-the-world on which any open membrane ends, by demonstrating that the necessary counterterms on such an open membrane reproduce the usual 10-dimensional gauge theory action.

In ref. [26] it was seen that T$-$duality not only naturally appears in the E_8 framework, but is forced upon the theory as the dimensional reduction becomes invalid when IIA is compactified on a circle of radius less than $\sqrt{\alpha\prime}$. The new circle, which appears in IIB, is seen to descend from the circle in the based loopgroup of the based loopgroup of E_8 which is fibred over the remaining 9 dimensions. The total space of the LLE_8 bundle includes both circles, providing a 12-dimension perspective (evidence is presented that these are the usual 12-dimensions of F-theory). In particular, a Calabi-Yau and its mirror appear to both be submanifolds of such a higher-dimensional manifold, as they differ by the exchange of these two circles. Thus it should be possible to translate any pair of calculation and mirror calculation into a single higher-dimensional calculation. We hope that this will allow a better understanding of mirror symmetry and F-theory. The possibility exists that other smooth topology changing transitions are also smooth in terms of the total space, but appear discontinuous because different dimensional reductions to 10-dimensions are valid before and after the transition.

Our final proposed calculation is one which has already been done, and will appear in the future. Combining the E_8 proposal with the above realization of T-duality, one may explicitly verify Hull's proposal for a geometric understanding of massive IIA [27], and connect it via two T-dualities to an alternate proposal [6] in which no dimensions are smaller than $\sqrt{\alpha\prime}$.

JHEP05(2003)048

5. Discussion

This paper is about an 11-dimensional E_8 gauge theory. One problem is that there are no 11-dimensional gauge theories. That is to say, the term $F \wedge *F$ is nonrenormalizable in 11-dimensions, and so we cannot have such a term in any microscopic description. The gauge theory may only be a low energy effective description, or alternatively, it does not contain the term $F \wedge *F$. In particular, the theory may be topological. A proposal for such a theory has been investigated by [12]. When compactified on an interval, at least in its simplest manifestation, it produces a dynamical 10d gauge theory on each boundary components satisfying the non-supersymmetric Fabinger-Horava [11] boundary conditions. A natural question is whether, via integrating out loops, the 11d SUGRA fields with the correct lagrangian (and so with supersymmetry) arise from some E_8 gauge theory.

The M-Theory partition function was originally shown to be well defined [2] using a mysterious E_8 bundle which restricts from 12 dimensions to the 11-dimensional bulk. We have tried to understand how the "existence" of such a bundle can be compatible with 11-dimensional supersymmetry.

An analogous construction to that of G_4 above may or may not exist for the elfbein or the graviton. This possibility is currently under investigation. Another tantalizing venue of future investigation is to investigate the link between this E_8 and that of the 11d $E_{8(8)}$ SUGRA of Nicolai and de Wit [8, 9]. The primary stumbling block to such a link is a factor of i in the decomposition of E_8 into the adjoint and spinor representations of SO(16), which leads to the noncompactness of the group in [8, 9]. To pursue such a program one must understand the role of this potentially devastating factor, beginning with the fact that it does not appear in ref. [28].

Acknowledgments

We would like to thank Allan Adams, Kurt Lechner, James Liu, Pieralberto Marchetti, Darius Sadri, Eric Sharpe, Uday Varadarajan and Jay Wacker for many useful questions and comments. In particular we would like to thank Jim and Uday for finding apocolyptic errors. This work was made possible by the good people of the INFN and by DOE grants DE-FG02-95ER40899 and DE-FG02-95ER40899.

References

[1] P. Hořava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys.* **B 475** (1996) 94 [hep-th/9603142].

[2] E. Witten, *On flux quantization in M-theory and the effective action*, *J. Geom. Phys.* **22** (1997) 1 [hep-th/9609122].

[3] D.-E. Diaconescu, G.W. Moore and E. Witten, E_8 *gauge theory and a derivation of K-theory from M-theory*, hep-th/0005090.

[4] C. Gomez and J.J. Manjarin, *Dyons, K-theory and M-theory*, hep-th/0111169.

[5] J. Evslin and U. Varadarajan, *K-theory and S-duality: starting over from square 3*, *J. High Energy Phys.* **03** (2003) 026 [hep-th/0112084].

[6] A. Adams and J. Evslin, *The loop group of E_8 and K-theory from 11D*, *J. High Energy Phys.* **02** (2003) 029 [hep-th/0203218].

[7] D. R. Morrison, *Half $K3$ surfaces*talk at Strings 2002, Cambridge, http://www.damtp.cam.ac.uk/strings02/avt/morrison/.

[8] B. de Wit and H. Nicolai, *Hidden symmetry in $D = 11$ supergravity*, *Phys. Lett.* **B 155** (1985) 47;
B. de Wit and H. Nicolai, *$D = 11$ supergravity with local* SU(8) *invariance*, *Nucl. Phys.* **B 274** (1986) 363.

[9] H. Nicolai, *Phys. Lett.* **B 187** (1987) 363.

[10] P. Hořava and E. Witten, *Heterotic and type-I string dynamics from eleven dimensions*, *Nucl. Phys.* **B 460** (1996) 506 [hep-th/9510209].

[11] M. Fabinger and P. Hořava, *Casimir effect between world-branes in heterotic M-theory*, *Nucl. Phys.* **B 580** (2000) 243 [hep-th/0002073].

[12] E. Diaconescu, D. Freed, and G. Moore, work in progress.

[13] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, *Phys. Lett.* **B 76** (1978) 409.

[14] M.F. Atiyah, V. Patodi, and I.M. Singer, *Spectral asymmetry and riemannian geometry 1*, *Math. Proc. Camb. Phil. Soc.* **77** (1975) 43; *Spectral asymmetry and riemannian geometry 2*, *Math. Proc. Camb. Phil. Soc.* **78** (1975) 405; *Spectral asymmetry and riemannian geometry 3*, *Math. Proc. Camb. Phil. Soc.* **79** (1976) 71.

[15] K. Hori, *Consistency condition for fivebrane in M-theory on R^5/Z_2 orbifold*, *Nucl. Phys.* **B 539** (1999) 35 [hep-th/9805141].

[16] A. Hanany and E. Witten, *Type-IIB superstrings, BPS monopoles and three-dimensional gauge dynamics*, *Nucl. Phys.* **B 492** (1997) 152 [hep-th/9611230].

[17] R.C. Myers, *Dielectric-branes*, *J. High Energy Phys.* **12** (1999) 022 [hep-th/9910053].

[18] A. Boyarsky, J.A. Harvey and O. Ruchayskiy, *A toy model of the M5-brane: anomalies of monopole strings in five dimensions*, *Ann. Phys. (NY)* **301** (2002) 1 [hep-th/0203154].

[19] E. Witten, *Five-brane effective action in M-theory*, *J. Geom. Phys.* **22** (1997) 103 [hep-th/9610234].

[20] D. Freed, J.A. Harvey, R. Minasian and G.W. Moore, *Gravitational anomaly cancellation for M-theory fivebranes*, *Adv. Theor. Math. Phys.* **2** (1998) 601 [hep-th/9803205].

[21] J.A. Harvey and O. Ruchayskiy, *The local structure of anomaly inflow*, *J. High Energy Phys.* **06** (2001) 044 [hep-th/0007037].

[22] K. Lechner, P.A. Marchetti and M. Tonin, *Anomaly free effective action for the elementary M5-brane*, *Phys. Lett.* **B 524** (2002) 199 [hep-th/0107061].

[23] P. Van Nieuwenhuizen, *Supergravity*, *Phys. Rept.* **68** (1981) 189.

[24] S. Sethi, unpublished.

[25] A. Adams, J. Evslin, and U. Varadarajan, to appear.

JHEP05(2003)048

[26] J. Evslin, *Twisted K-theory from monodromies*, hep-th/0302081.

[27] C.M. Hull, *Massive string theories from M-theory and F-theory*, *J. High Energy Phys.* **11** (1998) 027 [hep-th/9811021].

[28] T. Banks, N. Seiberg and E. Silverstein, *Zero and one-dimensional probes with $N = 8$ supersymmetry*, *Phys. Lett.* **B 401** (1997) 30 [hep-th/9703052].